

Absorption and scattering of a black hole with a $f(R)$ global monopole

M. A. Anacleto,^{1,*} F. A. Brito,^{1,2,†} S. J. S. Ferreira,^{1,‡} and E. Passos^{1,§}

¹*Departamento de Física, Universidade Federal de Campina Grande
Caixa Postal 10071, 58429-900 Campina Grande, Paraíba, Brazil*

²*Departamento de Física, Universidade Federal da Paraíba,
Caixa Postal 5008, 58051-970 João Pessoa, Paraíba, Brazil*

In this paper we consider the solution of a black hole with a global monopole in $f(R)$ gravity and apply the partial wave approach to compute the differential scattering cross section and absorption cross section. We show that in the low-frequency limit and at small angles the contribution to the dominant term in the scattering/absorption cross section is modified by the presence of the global monopole and the gravity modification. In such limit, the absorption cross section shows to be proportional to the area of the event horizon.

I. INTRODUCTION

Black holes are fascinating objects that have remarkable characteristics and one of them is that they behave like thermodynamic systems possessing temperature and entropy. Black holes are exact solutions of Einstein equations which are determined by mass (M), electric charge (Q) and angular momentum (J) [1, 2] and plays an important role in modern physics. In particular, black hole with a global monopole has been explored extensively by many authors in recent years [3–7] and the metric for this kind of black hole was determined by Barriola and Vilenkin [8]. The global monopoles are topological defects that arise in gauge theories due to the spontaneous symmetry breaking of the original global $O(3)$ symmetry to $U(1)$ [9, 10]. It is a type of defect that could be formed during phase transitions in the evolution of the early Universe. From the cosmological point of view, the so-called $f(R)$ theory of gravity introduces the possibility to explain the accelerated-inflation problem without the need to consider dark matter or dark energy [11–14]. In [15] it has been investigated the classical motion of a massive test particle in the gravitational field of a $f(R)$ global monopole. The authors in [16] have calculated, using the WKB approximation, the quasinormal modes for a black hole with a global monopole in $f(R)$ theory of gravity. The thermodynamics of the black holes with $f(R)$ global monopole was discussed in [17, 18] and was treated analytically in [19] the case of strong gravitational lensing for a massive source with a global monopole in $f(R)$ theory gravity. The main objective of this work is to compute the scattering cross section due to a black hole with a global monopole in $f(R)$ gravity theory. In [20] was analyzed the absorption problem for a massless scalar field propagating in general static spherically-symmetric black holes with a global monopole.

The study to understand the processes of absorption and scattering in the vicinity of black holes is one of the most important issues in theoretical physics and also of great relevance for experimental research. We can explore the dynamics of a black hole by trying to disturb it away from its stationary configuration. Thus, examining the interaction of fields with black holes is of great importance to understand aspects about formation, stability, and gravitational wave emission. For many years, several theoretical works have been done to investigate the black hole scattering [21] (see also references therein). Since 1970, many works have shown that at the long wavelength limit ($GM\omega \ll 1$) [22–29], the differential scattering cross section for small angles presents the following result: $d\sigma/d\Omega \approx 16G^2M^2/\theta^4$. In addition, the calculation to obtain the low energy absorption cross section has been studied extensively in the literature [30–33]. Thus, in this case the absorption cross section in the long-wavelength limit of a massless neutral scalar field is equal to the area of the horizon, $\sigma = 4\pi r_h^2 = 16\pi M^2$ [34]. On the other hand, for fermion fields one has been shown by Unruh [33] that the absorption cross section is $2\pi M^2$ in the low-energy limit. The result is exactly 1/8 of that for the scalar wave in the low-energy limit. An extension of the calculation of the absorption cross section for acoustic waves was performed in [35], [36] and [37]. The partial wave approach has also been extended to investigate the scattering by an acoustic black hole in $(2+1)$ dimensions [38–40] and also due to a non-commutative BTZ black hole [41]. Also some studies have been carried out on the processes of absorption and scattering of massive fields by black holes [42–46].

*Electronic address: anacleto@df.ufcg.edu.br

†Electronic address: fabrito@df.ufcg.edu.br

‡Electronic address: stefanejudith@gmail.com

§Electronic address: passos@df.ufcg.edu.br

In this paper, inspired by all of these previous works and adopting the technique developed by the authors in [38–41, 47], we shall focus on the computation of the scattering and absorption cross section for a monochromatic planar wave of neutral massless scalar field impinging upon a black hole with a $f(R)$ global monopole. In this scenario there are four parameters: the mass M of the black hole, the frequency ω of the field, the monopole parameter η and ψ_0 associated with the corrections from the $f(R)$ gravity. Thus, we have three dimensionless parameters: $GM\omega$, $8\pi G\eta^2 \approx 10^{-5}$ and $a = \omega/\psi_0$. In our analyzes, we will consider only the long-wavelength regime, in which $GM\omega \ll 1$. Dolan et al. [48], studied the analogous Aharonov-Bohm effect considering the scattering of planar waves by a draining bathtub vortex. They implemented an approximation formula to calculate the phase shift $\delta_l \approx (m - \tilde{m})$ analytically. In an analogous way we introduce the following approximation: $\delta_l \approx (l - \ell)$. Then, we have verified that the presence of the parameters η and ψ_0 modify the dominant term of the differential scattering cross section in the low-frequency limit at small angles and also the absorption cross section. We initially analyzed the example of the black hole with a global monopole and showed that the contribution to the dominant term of the differential cross section is increased due to the monopole effect as well as to the absorption. On the other hand, considering the case of a black hole with a global monopole in $f(R)$ theory, we find that the contribution to the dominant term of the differential scattering cross section is essentially due to the effect of the $f(R)$ theory. Here we adopt the natural units $\hbar = c = 1$.

II. DIFFERENTIAL SCATTERING CROSS SECTION

In this section we are interested in determining the differential scattering cross section for a black hole with a global monopole in $f(R)$ gravity by the partial wave method in the low frequency regime. For this purpose we will follow the procedure adopted in previous works to calculate the phase shift. Initially, we will consider a spherically symmetric line element of a black hole with a global monopole that is given by

$$ds^2 = A(r)dt^2 - \frac{dr^2}{A(r)} - r^2 d\Omega^2, \quad (1)$$

where

$$A(r) = 1 - 8\pi G\eta^2 - \frac{2GM}{r}. \quad (2)$$

Here, G is the Newton constant, η is the monopole parameter of the order 10^{16}GeV and so $8\pi G\eta^2 \approx 10^{-5}$ [8, 50]. The event horizon radius is obtained by $A(r) = 0$, i.e.

$$r_\eta = \frac{2GM}{(1 - 8\pi G\eta^2)} = \frac{r_h}{(1 - 8\pi G\eta^2)}, \quad (3)$$

where $r_h = 2GM$ is the event horizon of the Schwarzschild black hole.

The Hawking temperature of the black hole is

$$T_H = \frac{1}{4\pi} \left(\frac{1 - 8\pi G\eta^2}{r_h} \right). \quad (4)$$

For $\eta = 0$ the Hawking temperature of the Schwarzschild black hole is recovered.

The next step is to consider the Klein-Gordon wave equation for a massless scalar field in the background (1)

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi \right) = 0. \quad (5)$$

Now we can make a separation of variables into the equation above as follows

$$\Phi_{\omega lm}(\mathbf{r}, t) = \frac{R_{\omega l}(r)}{r} Y_{lm}(\theta, \phi) e^{-i\omega t}, \quad (6)$$

where ω is the frequency and $Y_{lm}(\theta, \phi)$ are the spherical harmonics.

In this case, the equation for $R_{\omega l}(r)$ can be written as

$$A(r) \frac{d}{dr} \left(A(r) \frac{dR_{\omega l}(r)}{dr} \right) + [\omega^2 - V_{eff}] R_{\omega l}(r) = 0, \quad (7)$$

and

$$V_{eff} = \frac{1}{r} \frac{dA(r)}{dr} + \frac{A(r)l(l+1)}{r^2}, \quad (8)$$

is the effective potential. At this point, we consider a new radial function, $\psi(r) = A^{1/2}(r)R(r)$, so we have

$$\frac{d^2\psi(r)}{dr^2} + U(r)\psi(r) = 0, \quad (9)$$

where

$$U(r) = \frac{[A'(r)]^2}{4A^2(r)} - \frac{A''(r)}{2A(r)} + \frac{\omega^2}{A^2(r)} - \frac{V_{eff}}{A^2(r)}, \quad (10)$$

and

$$A'(r) = \frac{dA(r)}{dr} = \frac{2GM}{r^2}, \quad A''(r) = \frac{d^2A(r)}{dr^2} = -\frac{4GM}{r^3}. \quad (11)$$

Now performing a power series in $1/r$ the Eq. (9) becomes

$$\frac{d^2\psi(r)}{dr^2} + [\tilde{\omega}^2 + \mathcal{V}(r) + \mathcal{U}(r)]\psi(r) = 0, \quad (12)$$

where now we have

$$\mathcal{V}(r) = \frac{4GM\tilde{\omega}^2}{(1-8\pi G\eta^2)r} + \frac{12\ell^2}{r^2}, \quad (13)$$

and

$$\begin{aligned} \mathcal{U}(r) = & \frac{32G^3M^3\tilde{\omega}^3 - 2(l^2+l)GM\tilde{\omega}(1-8\pi G\eta^2) - 16\pi G\eta^2GM\tilde{\omega}}{\tilde{\omega}(1-8\pi G\eta^2)^3r^3} \\ & + \frac{1}{\tilde{\omega}^2(1-8\pi G\eta^2)^4r^4} \left[80G^4M^4\tilde{\omega}^4 + G^2M^2\omega^2 - 4(l^2+l)G^2M^2\omega^2(1-8\pi G\eta^2) \right. \\ & \left. - (1-8\pi G\eta^2)[8-5(1-8\pi G\eta^2)]G^2M^2\tilde{\omega}^2 \right] + \dots, \end{aligned} \quad (14)$$

with $\tilde{\omega} = \omega/(1-8\pi G\eta^2)$ and

$$\ell^2 = -\frac{(l^2+l)}{12(1-8\pi G\eta^2)} + \frac{G^2M^2\tilde{\omega}^2}{(1-8\pi G\eta^2)^2}. \quad (15)$$

Notice that when $r \rightarrow \infty$ the potential $V(r) = \mathcal{V}(r) + \mathcal{U}(r) \rightarrow 0$ and the asymptotic behavior is satisfied. Thus, knowing the phase shifts the scattering amplitude can be obtained and which has the following partial-wave representation

$$f(\theta) = \frac{1}{2i\omega} \sum_{l=0}^{\infty} (2l+1) (e^{2i\delta_l} - 1) P_l \cos \theta, \quad (16)$$

and the differential scattering cross section can be computed by the formula

$$\frac{d\sigma}{d\theta} = |f(\theta)|^2. \quad (17)$$

The phase shift δ_l can be obtained applying the following approximation formula

$$\delta_l \approx \frac{1}{2}(l - \ell) = \frac{1}{2} \left(l - \sqrt{-\frac{(l^2+l)}{12(1-8\pi G\eta^2)} + \frac{G^2M^2\tilde{\omega}^2}{(1-8\pi G\eta^2)^2}} \right). \quad (18)$$

In the limit $l \rightarrow 0$ we obtain

$$\delta_l = -\frac{GM\tilde{\omega}}{2(1-8\pi G\eta^2)} + \mathcal{O}(l) = -\frac{GM\omega}{2(1-8\pi G\eta^2)^2} + \mathcal{O}(l). \quad (19)$$

Note that in the limit $l \rightarrow 0$ the phase shifts tend to non-zero term, which naturally leads to a correct result for the differential cross section at the small angles limit. Another way of obtaining the same phase shift is through the Born approximation formula

$$\delta_l \approx \frac{\omega}{2} \int_0^\infty r^2 J_l^2(\omega r) \mathcal{U}(r) dr, \quad (20)$$

where $J_l(x)$ are the spherical Bessel functions of the first kind and $\mathcal{U}(r)$ is the effective potential of Eq. (14). After performing the integration we take the limits of $\omega \rightarrow 0$ and $l \rightarrow 0$. So the result is the same as Eq. (19).

The Eq. (16) is poorly convergent, so it is very difficult to perform the sum of the series directly. This is due to the fact that an infinite number of Legendre polynomials are required to obtain divergences in $\theta = 0$. In [51], it has been found by the authors a way to around this problem. It has been proposed by them a reduced series which is less divergent in $\theta = 0$, i.e.

$$(1 - \cos \theta)^m f(\theta) = \sum_{l=0} a_l^m P_l \cos \theta, \quad (21)$$

and so it is expected that the reduced series can converge more quickly.

Therefore, to determine the differential scattering cross section, we will use the following equation [51, 52]

$$\frac{d\sigma}{d\theta} = \left| \frac{1}{2i\omega} \sum_{l=0}^1 (2l+1) (e^{2i\delta_l} - 1) \frac{P_l \cos \theta}{1 - \cos \theta} \right|^2. \quad (22)$$

However, considering few values of l ($l = 0, 1$) is sufficient to obtain the result satisfactorily. Hence the differential scattering cross section is in this case given by

$$\left. \frac{d\sigma}{d\theta} \right|_{\omega \rightarrow 0}^{\text{lf}} = \frac{16G^2 M^2}{(1 - 8\pi G\eta^2)^4 \theta^4} + \dots = \frac{16G^2 M^2}{\theta^4} \left[1 + 32\pi G\eta^2 + \mathcal{O}(G\eta^2)^2 \right] + \dots. \quad (23)$$

The dominant term is modified by monopole parameter η . Thus, we verified that the differential cross section is increased by the monopole effect. As $\eta = 0$ we obtain the result for the Schwarzschild black hole case.

We will now compute the differential scattering cross section of a black hole with a global monopole in the $f(R)$ gravity. The spherical symmetric line element is given as follow [15, 18, 49]

$$ds^2 = A(r)dt^2 - \frac{dr^2}{A(r)} - r^2 d\Omega^2, \quad (24)$$

where

$$A(r) = 1 - 8\pi G\eta^2 - \frac{2GM}{r} - \psi_0 r. \quad (25)$$

The term $\psi_0 r$ corresponds to the extension of the standard general relativity. For metric (24), when $A(r) = 0$ we have a horizon of events

$$r_h = \frac{1 - 8\pi G\eta^2 - \sqrt{(1 - 8\pi G\eta^2)^2 - 8GM\psi_0}}{2\psi_0}, \quad (26)$$

and a cosmological horizon

$$r_C = \frac{1 - 8\pi G\eta^2 + \sqrt{(1 - 8\pi G\eta^2)^2 - 8GM\psi_0}}{2\psi_0}. \quad (27)$$

Note that the cosmological horizon exists only if ψ_0 is nonzero. Considering that ψ_0 is small and expanding the root term in Eqs. (26) and (27) we obtain

$$r_h \approx 2GM, \quad (28)$$

that is the event horizon of the Schwarzschild black hole and for Eq. (27) we find

$$r_{\psi_0} = \frac{1}{\psi_0} - \frac{8\pi G\eta^2}{\psi_0} + \dots \approx \frac{1}{\psi_0}. \quad (29)$$

The Hawking temperature associated with the black hole of Eq.(24) is

$$T_H = \frac{1}{4\pi} \left(\frac{1 - 8\pi G\eta^2}{r_h} - 2\psi_0 \right). \quad (30)$$

If $\eta = 0$ and $\psi_0 = 0$ (in the absence of monopoles and $f(R)$ corrections) the Hawking temperature will be reduced to that of the Schwarzschild case, as expected.

Following the same steps applied to the previous case, Eq. (9) can be now written as

$$\frac{d^2\psi(r)}{dr^2} + \left[\frac{(\ell^2 + 1/4)}{r^2} + U(r) \right] \psi(r) = 0, \quad (31)$$

being

$$U(r) = \frac{a(1 - 8\pi G\eta^2)/2 + a(\ell^2 + l + 1) + 2(1 - 8\pi G\eta^2)a^3}{\omega r^3} + \frac{1}{\omega^2 r^4} \left[-4GM\omega(a^3 + a) + (3 - 48\pi G\eta^2)a^4 \right. \\ \left. + \frac{3a^2}{4} - 8\pi G\eta^2 a^2 + a^2 l(l + 1)(1 - 8\pi G\eta^2) + 2a^2(1 - 8\pi G\eta^2) \right] + \dots, \quad (32)$$

where we have defined $\ell^2 = a^2$ and $a = \omega/\psi_0$. Note that the potential $V(r) = (\ell^2 + 1/4)/r^2 + U(r)$ obeys the asymptotic limit $V(r) \rightarrow 0$ as $r \rightarrow \infty$.

Next using the approximation formula (18) the phase shift δ_l in the limit $l \rightarrow 0$ reads

$$\delta_l = -\frac{\omega}{2\psi_0} + \mathcal{O}(l). \quad (33)$$

Also in this case the phase change tends to a non-zero constant term in the limit $l \rightarrow 0$. Once again we can verify that the phase shift could have been obtained from the Born approximation formula

$$\delta_l \approx \frac{\omega}{2} \int_0^\infty r^2 J_l^2(\omega r) U(r) dr, \quad (34)$$

Thus, in the low-frequency (long-wavelength) limit and at the small angle θ , the differential scattering cross section is given by

$$\left. \frac{d\sigma}{d\theta} \right|_{\omega \rightarrow 0}^{\text{lf}} = \left| \frac{1}{2i\omega} \sum_{l=0}^1 (2l+1) (e^{2i\delta_l} - 1) \frac{P_l \cos \theta}{1 - \cos \theta} \right|^2 = \frac{16}{\psi_0^2 \theta^4} + \dots. \quad (35)$$

We see that the presence of the parameters ψ_0 modifies the dominant term.

III. ABSORPTION CROSS SECTION

In this section we will determine the absorption cross section for a black hole with a global monopole in $f(R)$ gravity in the low-frequency limit. As is well known in quantum mechanics, the total absorption cross section can be computed by means of the following relation

$$\sigma_{abs} = \frac{\pi}{\omega^2} \sum_{l=0}^{\infty} (2l+1) \left(|1 - e^{2i\delta_l}|^2 \right). \quad (36)$$

For the phase shift δ_l of the Eq. (19), we obtain in the limit $\omega \rightarrow 0$:

$$\sigma_{abs}^{\text{lf}} = \frac{\pi}{\omega^2} \sum_{l=0}^3 (2l+1) \left(|1 - e^{2i\delta_l}|^2 \right) = \frac{16\pi G^2 M^2}{(1 - 8\pi G\eta^2)^4}, \\ = \frac{\mathcal{A}_{Sch}}{(1 - 8\pi G\eta^2)^4}, \quad (37)$$

where $\mathcal{A}_{Sch} = 4\pi r_h^2$ is the area of the event horizon of the Schwarzschild black hole. So for a few values of l ($l = 0, 1, 2, 3$) the result is successfully obtained. Here we note that the absorption is increased due to the contribution of the monopole.

Now for the phase shift δ_l (33) and applying the limit $\omega \rightarrow 0$ we find

$$\sigma_{abs}^{lf} = \frac{\pi}{\omega^2} \sum_{l=0}^3 (2l+1) \left(|1 - e^{2i\delta_l}|^2 \right) = \frac{16\pi}{\psi_0^2} = 16\pi r_{\psi_0}^2 = 4\mathcal{A}_{\psi_0}. \quad (38)$$

Therefore our results for absorption shows concordance with the universality property of the absorption cross section which is always proportional to the area of the event horizon at low-frequency limit [53].

IV. CONCLUSIONS

In summary, in the present study we calculate the absorption and scattering cross section of a black hole with a $f(R)$ global monopole in the low-frequency limit at small angles ($\theta \approx 0$). To determine the phase shift analytically we have implemented the approximation formula $\delta_l \approx (l - \ell)$ and so we have found, adopting the partial wave approach, that the scattering cross section is still dominated at the small-angled limit by $1/\theta^4$. This dominant term is modified by the presence of the parameters η and ψ_0 . Initially the case of a black hole with a global monopole was analyzed and we showed that the result for the differential scattering cross section as well as the absorption cross section is increased due to the monopole effect. Moreover, considering the case of a black hole with a $f(R)$ global monopole, we find that the contribution to the dominant term of the differential scattering cross section is essentially due to the effect of the $f(R)$ gravity. We also show that the absorption cross section is proportional to the area of the event horizon at low-frequency limit.

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